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On the edge double Roman domination number of planar graph

Mina Valinavaz[a,*]

[a]Department of Mathematics, Azarbaijan Shahid Madani University Tabriz, I.R. Iran.

Abstract

An *edge double Roman dominating function* (EDRDF) on a graph G is a function $f : E(G) \to \{0, 1, 2, 3\}$ satisfying the condition that such that every edge e with $f(e) = 0$, is adjacent to at least two edge e, e' for which $f(e) = f(e') = 2$ or one edge e'' with $f(e'') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geqslant 2$. The Edge double Roman dominating number of G, denoted by $\gamma'_{dR}(G)$, is the minimum weight $w(f) = \sum_{e \in E(G)} f(e)$ of an edge double Roman dominating function f of G. In this paper, we introduction some results on the edge double Roman domination number of a graph. Also, we provide some upper and lower bounds for the edge double Roman domination number of graphs.

Keywords: Double Roman dominating function, Double Roman domination number, Edge double Roman dominating function, Edge double Roman domination number.

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. A graph G is k-*regular* if $d(v) = k$ for each vertex v of G. A *leaf* is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf, and a *strong support vertex* is a support vertex adjacent to at least two leaves. An edge incident to a leaf is called a *pendant edge*. A *tree* is an acyclic connected graph. A tree T is a *double star* if it contains exactly two vertices that are not leaves. For $a, b \geqslant 2$, a double star whose support vertices have degree a and b is denoted by $S(a, b)$. If T is a rooted tree, we for each vertex v, we denote by T_v the sub-rooted tree rooted at v. The height of a rooted is the maximum distance from the root to a leaf.

The complement of a graph G is denote by $\overline{G}$. We write K_n for the complete graph of order n, C_n for the cycle of length n, and P_n for the path of order n. A *matching* is any independent set of edges. A maximal matching is a matching X so that $V(X) - V(G)$ is an independent set of vertices. A *perfect matching* in graph G is a matching so that $V(X) = V(G)$. The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ee' \in E(L(G))$ when $e = uv$ and $e' = vw$ in G. It is easy to see that $L(K_{1,n}) = K_n$, $L(C_n) = C_n$ and $L(P_n) = P_{n-1}$. For a subset S of vertices of G, and a vertex $x \in S$,

*Corresponding author
Email address: m.valinavaz@azaruniv.ac.ir (Mina Valinavaz)

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we may that a vertex $y \notin S$ is a *private neighbor* of x with respect to S if $N(y) \cap S = \{x\}$.
A *double Roman dominating function* on a graph G is defined by Beeler, Haynes and Hedetniemi in [7] as a function $f : V \longrightarrow \{0,1,2,3\}$ having the property that if $f(u) = 0$, then vertex u has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(u) = 1$, then vertex u must have at least one neighbor w with $f(w) \geqslant 2$. The *weight*, $\omega(f)$, of f is defined as $f(V(G))$.

A *Edge double Roman dominating function*(EDRDF) of graph G is a function $f : E(G) \longrightarrow \{0,1,2,3\}$ having the property that if $f(e) = 0$, then edge e has at least two neighbors assigned 2 under f or one neighbor e' with $f(e') = 3$, and if $f(e) = 1$, then edge e must have at least one neighbor e' with $f(e') \geqslant 2$. The *weight* of an edge double Roman dominating number of f, denote by $\omega(f)$, is the value $\sum_{e\in E(G)} f(e)$. The weight of a EDRDF, $\sum_{e\in E(G)} f(e)$. The minimum weight of a EDRDF is the edge double roman domination number of G, denoted by $\gamma'_{dR}(G)$. If f is a EDRDF in a graph G, then we simply can represent f by $f = (E_0, E_1, E_2, E_3)$(or $f = (E_0^f, E_1^f, E_2^f, E_3^f)$ to refer to f), where $E_0 = \{e \in E(G) : f(e) = 0\}, E_1 = \{e \in E(G) : f(e) = 1\}, E_2 = \{e \in E(G) : f(e) = 2\}$, and $E_3 = \{e \in E(G) : f(e) = 3\}$.
In this note we initiate the study of the Edge double Roman domination in graphs and present some (sharp) bounds for this parameter. In addition, we determine the Edge double Roman domination number of some classes of graphs.

2. Trees

Theorem 2.1. *If T is a tree with order $n \geqslant 2$, then $\gamma'_{dR}(T) \leqslant n$.*

Proof. Let T be a tree with order $n \geqslant 2$, we will proceed by induction on n. If $n = 2$, then theorem is obvious. If $diam(T) = 2$ or 3, then T has a dominating edge, and $\gamma'_{dR}(G) = 3 \leqslant n$. Hence, we may assume that the diameter of T is at least 4. This implies that $n \geqslant 5$.
Assume that any tree T' with order n' has $\gamma'_{dR}(T) \leqslant n'$. Among all longest paths in T. Chose P to be one that maximizes the degree of its next-to-last vertex v. Let u be the parent of v, that is, u is the non-leaf neighbor of v, and let w be a leaf neighbor of v. Note that by our choice of v, every child of v is a leaf. We consider three cases.
Case 1. $deg_T(v) > 2$. Then v has a least two leaf children. Consider $T' = T - T_v$. Define f on $E(G)$ by letting $f(vw) = 3$ and 0 for each neighbor edges of v except vw. Then f is an edge double Roman dominating function of T, implying that $\gamma'_{dR}(T) \leqslant w(f) = w(f') \leqslant \gamma'_{dR}(T') \leqslant n' = (n-1) < n$.
Case 2.$deg_T(v) = deg_T(u) = 2$, obtain T' by deleting u, v and w. Let z be other neighbor of u. Define f on $E(G)$ by letting $f(x) = f'(x)$, except for $f(zu) = f(vw) = 0$ and $f(uv) = 3$. Again f is an EDRDF for T and that $\gamma'_{dR}(T) \leqslant w(f) = w(f') + 3 \leqslant \gamma'_{dR}(T') + 3 \leqslant n' + 3 = (n-3) + 3 = n$.
Case 3. $deg_T(u) > 2$ and let k be the number of children of u, other v, with degree 2 and s be the number of leaf children of u in T'. $T' = T - T_u$. Define f on $V(T)$ by $f(x) = f'(x)$, except for $f(uv) = 3$ and $f(x) = 2$ for each non-neighbor edges x of uv outside T'' except vw. Again f is an EDRDF we have $\gamma'_{dR}(T) \leqslant w(f) = w(f') + 2k + 1 \leqslant \gamma'_{dR}(T') + 2k + 1 \leqslant n' + 2k + 1 = (n - (2k + s + 1)) + 2k + 1 = n - s \leqslant n$. □

Theorem 2.2. *For a tree T of order n, $\gamma'_{dR}(T) \geqslant n - \ell(T) + 1$*

Proof. Let T be a tree with order $n \geqslant 2$, we will proceed by induction on n. If $n = 2$or $diam(T) = 2$ or 3, theorem is obvious. Hence, we may assume that the diameter of T is at least 4. This implies that $n \geqslant 5$.
Assume that any tree T' with order n' has $\gamma'_{dR}(T) \leqslant n'$. Among all longest paths in T. Chose P to be one that maximizes the degree of its next-to-last vertex v. Let u be the parent of v, that is, u is the non-leaf neighbor of v, and let w be a leaf neighbor of v. Note that by our choice of v, every child of v is a leaf. We consider two cases.

Case 1. $\deg_T(v) > 2$. Consider $T' = T - w$. T' is an induced subgraph of T and induction hypothesis, $n - \ell(T) + 1 = n - 1 - \ell(T') + 1 \leqslant \gamma'_{dR}(T') \leqslant \gamma'_{dR}(T)$.
Case 2. $\deg_T(v) = 2$, $\deg_T(u) > 2$ and let k be the number of children of u, other v, with degree 2 and s be the number of leaf children of u in T'. $T' = T - T_u$. Define f on $V(T)$ by $f(x) = f'(x)$, except for $f(uv) = 3$ and $f(x) = 2$ for each non-neighbor edges x of uv outside T'' except vw. Again f is an EDRDF we have $w(f) = 2k + 1 + w(f') \geqslant 2k + 1 + (n - k - s - \ell(T') + 1) \geqslant 2k + 1 + (n - 2k - s - 1 - (\ell(T) - k - s + 1) + 1) \geqslant n - \ell(T) + k \geqslant n - \ell(T) + 1$.

□

3. edge double Roman domination number of planar graphs

A graph is *Outerplanar* if it can be embedded in the plane such that every vertex is incident to the infinite face.

Lemma 3.1. *[16] Every simple outerplanar graph has a vertex of degree at most* 2.

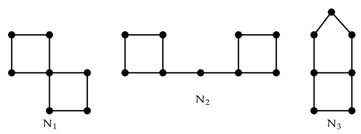

Figure 1: h

Theorem 3.2. *If* G *is an outerplanar graph of order* n, *then*

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{6n}{5} \rfloor,$$

and this bound is sharp for such graphs $C_3, C_7, C_5, N_1, N_2, N_3$.

Proof. We prove the assertion by induction on n. If $n = 1$ or 2, then theorem is obvious. So assume that $n \geqslant 3$ and $\mathrm{diam}(G) \geqslant 3$. By lemma3.1, $\delta(G) \leqslant 2$. Let $v \in V(G)$ has minimum degree. There are two cases.
Case 1. $\delta(G) = 1$.
Let v a vertex of degree 1 and $vw \in E(G)$. Since $\mathrm{diam}(G) \geqslant 3$, we can assume that the vertex w has another neighbor u. Let $G' = G \backslash \{v, w, u\}$. By induction hypothesis G' has a γ'_{dR}-function, say g', whose weight is at most $\lfloor \frac{6(n-3)}{5} \rfloor$. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = wu \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{6(n-3)}{5} \rfloor + 3 < \lfloor \tfrac{6n}{5} \rfloor.$$

Case 2. $\delta(G) = 2$.
Let v a vertex of degree 2 in G and $N_G(v) = \{u, u'\}$.If $uu' \in E(G)$, then let $G' = G \backslash \{v, u, u'\}$. By induction hypothesis G' has a γ'_{dR}-function, say g', whose weight is at most $\lfloor \frac{6(n-3)}{5} \rfloor$. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = uu' \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor \frac{6(n-3)}{5} \rfloor + 3 < \lfloor \frac{6n}{5} \rfloor.$$

Thus assume that $uu' \notin E(G)$ and the vertex u has a neighbor w different from u, u'. If u' has a neighbor w' different from u, v, w, then let $G' = G \backslash \{v, u, u', w, w'\}$. By induction hypothesis G' has a γ'_{dR}-function, say g', whose weight is at most $\lfloor \frac{6(n-5)}{5} \rfloor$. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = uw, u'w' \\ 0 & \quad \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor \frac{6(n-5)}{5} \rfloor + 6 = \lfloor \frac{6n}{5} \rfloor.$$

If the vertex u' has not a neighbor different from v, w, then since $\delta(G) = 2$, $d(u') = 2$ and we have $u'w \in E(G)$. Similar before, let $G' = G \backslash \{v, u, u', w\}$. By induction hypothesis G' has a γ'_{dR}-function, say g', whose weight is at most $\lfloor \frac{6(n-3)}{5} \rfloor$. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = uw \\ 2 & \text{if} \quad e = vu' \\ 0 & \quad \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor \frac{6(n-4)}{5} \rfloor + 5 < \lfloor \frac{6n}{5} \rfloor.$$

The proof is complete. □

Lemma 3.3. *If G is a planar graph of girth at least 6, then $\delta(G) \leqslant 2$.*

Proof. Suppose that G has order n and size e and f is the number its faces. Assume that $\delta(G) \geqslant 3$. So $e \geqslant \frac{3}{2}n$. on the otherhand, since $g(G) \geqslant 6$, $6f \leqslant 2e$. By using Euler Formula for planar graphs we have

$$\begin{aligned} n &= 2 + e - f \\ &> 2 + e - \frac{e}{3} \\ &> 2 + \frac{2e}{3} \\ &> 2 + n. \end{aligned}$$

a contradiction. □

We have the some theorem for planar graphs with girth at least 6.

Theorem 3.4. *If G is a planar graph of order n with girth at least 6, then $\gamma'_{dR}(G) \leqslant \lfloor \frac{6n}{5} \rfloor$.*

The graph G is called *claw-free* if G has not $K_{1,3}$ as an induced subgraph.

Lemma 3.5. *[16] If G is a planar graph, then $\delta(G) \leqslant 5$.*

Theorem 3.6. *If G is a planar claw-free graph of order, then*

$$\gamma'_{dR}(G) \leqslant \lfloor \frac{5n}{4} \rfloor,$$

and this bound is sharp for the graph S.

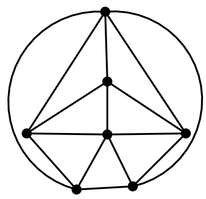

Figure 2: graph S

Proof. We follow the idea of the proof of theorem 3.2 and prove the assertion by induction on n. The assertion is true when $n = 1$ or 2 or $\mathrm{diam}(G) = 2$ or 3. Let v be a vertex of G with minimum degree. By lemma 3.5, $\delta(G) \leqslant 5$ and we have the following five cases.

Case 1. $\delta(G) = 1$.

Let G' and g' be that one introduce in Case 1 of the proof of theorem 3.2. By induction hypothesist we have

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{5(n-3)}{4} \rfloor + 3 < \lfloor \tfrac{5n}{4} \rfloor.$$

Case 2. $\delta(G) = 2$.

Consider the notations of Case 2 of the proof of theorem 3.2. If $uu' \in E(G)$, then

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{5(n-3)}{4} \rfloor + 3 < \lfloor \tfrac{5n}{4} \rfloor.$$

Assume that $uu' \notin E(G)$ and the vertex u has a neighbor w different from v, u'. If u' has a neighbor w' different from v, u, w, then

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{5(n-5)}{4} \rfloor + 6 < \lfloor \tfrac{5n}{4} \rfloor.$$

If the vertex u' has not a neighbor different than v, w, then

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{5(n-4)}{4} \rfloor + 3 < \lfloor \tfrac{5n}{4} \rfloor.$$

Case 3.$\delta(G) = 3$.

Let $N(v) = \{u_1, u_2, u_3\}$. Since G is claw-free, we can assume that $u_1u_2 \in E(G)$. Since $\delta(G) = 3$ and $\mathrm{diam}(G) \geqslant 3$, we can assume that there exists $w \in N(u_3)\backslash\{v, u_1, u_2\}$. Let $G' = G\backslash\{v, u_1, u_2, u_3, w\}$. By induction hypothesis G' has a γ'_{dR}-function, say g', whose weight is at most $\lfloor \frac{5(n-5)}{4} \rfloor$. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = u_1u_2, u_3w \\ 0 & \quad\;\; \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor \tfrac{5(n-5)}{4} \rfloor + 6 < \lfloor \tfrac{5n}{4} \rfloor.$$

Case 4. $\delta(G) = 4$.

Let $N(v) = \{u_1, u_2, u_3, u_4\}$. Since G is claw-free, we can assume that $u_1u_2 \in E(G)$. If $u_3u_4 \in E(G)$, then let $G' = G\backslash\{v, u_1, u_2, u_3, u_4\}$. By induction hypothesis G' has a γ'_{dR}-function, say g', whose weight is at most $\lfloor \frac{5(n-5)}{4} \rfloor$. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = u_1u_2, e = u_3u_4 \\ 0 & \quad\;\; \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor\frac{6(n-5)}{5}\rfloor + 6 < \lfloor\frac{5n}{4}\rfloor.$$

So assume that $u_3u_4 \notin E(G)$. Since $\delta(G) = 4$, u_3 and u_4 have neighbors w and w' different from u_1, u_2, v, respectively if $w \neq w'$, then let $G' = G\backslash\{v, u_1, u_2, u_3, u_4, w, w'\}$ and g' be an EDRDF for G'. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if } \ e \in E(G') \\ 3 & \text{if } \ e = u_1u_2, e = u_3w, u_4w' \\ 0 & \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor\frac{5(n-7)}{4}\rfloor + 9 < \lfloor\frac{5n}{4}\rfloor.$$

Otherwise, $N(u_3) = N(u_4) = \{v, u_1, u_2, w\}$ and this contradicts. The fact that G is planar and has no subdivision of K_5 as a subgraph.

Case 5. $\delta(G) = 5$.

Let $N(v) = \{u_1, u_2, u_3, u_4, u_5\}$. Since G is claw-free, we can assume that $u_1u_2, u_3u_4 \in E(G)$.
If $w \in N(u_5)\backslash\{v, u_1, u_2, u_3, u_4\}$, then let $G' = G\backslash\{v, u_1, u_2, u_3, u_4, u_5, w\}$ and g' be an EDRDF for G'. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if } \ e \in E(G') \\ 3 & \text{if } \ e = u_1u_2, e = u_3u_4, e = u_5w \\ 0 & \text{otherwise} \end{cases}$$

The function g is an EDRDF. Therefore, we find that

$$\gamma'_{dR}(G) \leqslant \lfloor\frac{5(n-7)}{4}\rfloor + 9 < \lfloor\frac{5n}{4}\rfloor.$$

Otherwise, $N(u_5) = \{v, u_1, u_2, u_3, u_4\}$. By switching rolls of u_3 and u_5 we can assume that $N(u_3) = \{v, u_1, u_2, u_4, u_5\}$ and this contradicts. The fact that G is planar and has no subdivision of K_5 as a subgraph. □

The graph G is called *triangle-free*, if G is not K_3 as an induced subgraph.

Lemma 3.7. *If G is planar triangle-free graph, then $\delta(G) \leqslant 3$.*

Proof. Suppose that G has order n and size e and f is the number its faces. Assume that $\delta(G) \geqslant 4$. So $e \geqslant 2n$. on the otherhand, since $g(G) \geqslant 4$, $4f \leqslant 2e$. By using Euler Formula for planar graphs we have

$$\begin{aligned} n &= 2 + e - f \\ &> 2 + e - \tfrac{e}{2} \\ &> 2 + \tfrac{e}{2} \\ &> 2 + n. \end{aligned}$$

a contradiction. □

Theorem 3.8. *If G is a planar triangle-free graph of order n, Then*

$$\gamma'_{dR}(G) \leqslant \lfloor\frac{6n}{5}\rfloor.$$

Proof. We follow the idea of the proof of theorem 3.2 and prove the assertion by induction on n. The assertion is true when $n = 1$ or 2 or $diam(G) = 2$ or 3. Let v be a vertex of G with minimum degree. By lemma 3.7, $\delta(G) \leqslant 3$ and we have the following five cases.

Case 1. $\delta(G) \leqslant 2$.

Similar case 1 and 2 of the proof of theorem 3.2. By induction hypothesist we have

$$\gamma'_{dR}(G) < \lfloor \frac{6n}{5} \rfloor.$$

Case 2.$\delta(G) = 3$.
Let $N(v) = \{u_1, u_2, u_3\}$. Since G is triangle-free and $\delta(G) = 3$, we can assume that u_1u_2 and u_3 has distinct neighbors w, w' and w'' in $V(G)\setminus\{v, u_1, u_2, u_3\}$, respectively. Let $G' = G\setminus\{v, u_1, u_2, u_3, w, w', w''\}$ and G'be an EDRDF for G'. Define $g : E(G) \to \{0, 1, 2, 3\}$

$$g(e) = \begin{cases} g'(e) & \text{if} \quad e \in E(G') \\ 3 & \text{if} \quad e = u_1w, e = u_2w', e = u_3w'' \\ 0 & \quad\text{otherwise} \end{cases}$$

The function g is an EDRDF and so by induction hypothesis

$$\gamma'_{dR}(G) \leqslant \lfloor \frac{6(n-7)}{5} \rfloor + 9 < \lfloor \frac{6n}{5} \rfloor.$$

□

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